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Rational Approximation and n-Dimensional Diameter*

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1. INTRODUCTION

Any function, analytic on an open set Ω in the extended complex plane, can be uniformly approximated by rational functions on each closed subset of Ω. This is one version of the Runge approximation theorem. There arises naturally the question of how well such approximations can be effected. It is this question, and some related matters, that form the subject of this paper.

To set things up precisely, let A denote the set of those functions f which are analytic in Ω and satisfy $|f| \leqslant 1$ there. For each $f \in A$, and for each closed set $K \subset \Omega$, let

$$r_n(f) = \inf_R \max_{z \in K} |f(z) - R(z)|, \tag{1}$$

where R runs through all rational functions of degree (number of poles in the extended plane) at most n. Finally, let

$$r_n = \sup_{f \in A} r_n(f).$$

Thus, r_n is a measure of how well the functions in A can be approximated by rational functions of degree n.

Runge's theorem asserts that the rational functions of approximation may be taken to have poles lying outside Ω. Thus, we also define quantities $\tilde{r}_n(f)$ and $\tilde{r}_n$, analogous to $r_n(f)$ and r_n, but where the rational functions R appearing in (1) are required to have their poles outside Ω.

Another measure of how well functions of A can be approximated on K

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is the n-dimensional diameter of A considered as a subset of the space of continuous functions on K. This is defined as

$$d_n = \inf_{E_n} \sup_{f \in A} \operatorname{dist}(f, E_n).$$

Here E_n runs through all n-dimensional linear spaces of continuous functions on K and the distance is taken in the metric

$$\operatorname{dist}(f, g) = \max_{z \in K} |f(z) - g(z)|. \tag{2}$$

Thus d_n is a measure of how well functions of A can be approximated on K by functions from some n-dimensional space of continuous functions. We shall see that for n large a certain space of rational functions gives almost as good approximations as any space.

The main result of this paper is that with some mild restrictions on K and Ω we have

$$\lim_{n\to\infty} r_n^{1/n} = \lim_{n\to\infty} \tilde{r}_n^{1/n} = \lim_{n\to\infty} d_n^{1/n} = e^{-1/C(K,\Omega)}$$

where $C(K, \Omega)$ is the Green capacity of K relative to Ω. This capacity is defined to be least upper bound of $\mu(K)$ for all nonnegative Borel measures μ supported on K and satisfying

$$\int g(z, \zeta)\, d\mu(\zeta) \leqslant 1, \qquad z \in \Omega.$$

Here $g(z, \zeta)$ is Green's function for Ω.

We should mention that $g(z, \zeta)$ exists only if $\partial\Omega$ has positive (logarithmic) capacity. However if $\partial\Omega$ has zero capacity then A consists only of constant functions (any bounded harmonic function on Ω may be extended to be harmonic on the entire extended plane [7, Theorem III.29] and so is constant), so for any K we have $r_n = \tilde{r}_n = d_n = 0$. We may assume therefore that $\partial\Omega$ has positive capacity.

Particular cases have been previously established in different forms. Erokhin [2] showed that if Ω is simply connected and K is a continuum then $\lim d_n^{1/n}$ is the reciprocal of the modulus of the doubly connected domain Ω–K. In certain cases where Ω and K are bounded by finitely many analytic Jordan curves Levin and Tichomirov [5] expressed $\lim d_n^{1/n}$ and $\lim \tilde{r}_n^{1/n}$ in terms of the flux across appropriate curves of the harmonic measure of K with respect to Ω–K. See also [8, Chap. IX] for upper bounds on $\tilde{r}_n$ in certain cases.

The limiting behavior of d_n for large n gives an asymptotic formula for the entropy of A with respect to the metric (2). The ϵ-entropy $H_\epsilon(A)$ is the logarithm of the smallest number of sets of diameter at most 2ϵ which cover A.

Then

$$\lim_{n\to\infty} d_n^{1/n} = e^{-1/C(K,\Omega)}$$

implies

$$\lim_{\epsilon\to 0} \frac{H_\epsilon(A)}{(\log \epsilon^{-1})^2} = C(K, \Omega).$$

See [5, section 5] or [6, p. 164].

The lower bound for d_n is obtained, interestingly enough, by showing that a certain Riemann boundary value problems has many solutions. As far as we know this is the first application of the Riemann problem to approximation theory.

Another related matter is the following. Let K_1 and K_2 be arbitrary disjoint closed sets (of positive logarithmic capacity) in the extended plane, and set

$$s_n = \sup_R \frac{\min\{|R(z)|: z \in K_1\}}{\max\{|R(z)|: z \in K_2\}}$$

where the supremum is taken over all rational functions of degree n. Let K_1' denote the complement of K_1. Then it is known that

$$s_n \leqslant e^{n/C(K_2, K_1')} \tag{3}$$

(See [4] for the case where K_1 and K_2 are continua and the inequality stated in a different form; see [9] for the general case.) We shall show here that the constant $C(K_2, K_1')$ is best possible in the sense that

$$\lim_{n\to\infty} s_n^{1/n} = e^{1/C(K_2, K_1')}. \tag{4}$$

2. Some Basic Facts

We list here some facts about the capacities $C(K, \Omega)$ that we shall need later.

(a) There is an equilibrium measure μ on K satisfying

$$\mu(K) = 1$$

$$u_\mu(z) = \int g(z, \zeta)\, d\mu(\zeta) \leqslant 1/C(K, \Omega), \qquad z \in \Omega.$$

(b) This μ is supported on $\partial\Omega$ and $u_\mu = 1/C(K, \Omega)$ on each open subset of K.

(c) If $\{K_n\}$ is either a decreasing sequence of closed sets with intersection K or an increasing sequence of closed sets with union K, then

$$\lim_{n\to\infty} C(K_n, \Omega) = C(K, \Omega).$$

(d) If $\{\Omega_n\}$ is either an increasing sequence of open sets with union Ω or a decreasing sequence of open sets with intersection Ω, then

$$\lim_{n\to\infty} C(K, \Omega_n) = C(K, \Omega).$$

Facts (a)–(c) are well-known properties of general capacities. (See, for example, [1].) Moreover, we shall show below (see the remark following Lemma 4 in section 5) that if K' is the complement of K and Ω' of Ω, then

$$C(K, \Omega) = C(\Omega', K').$$

Thus (d) follows from (c).

3. Upper Bounds

In this section we obtain upper bounds for r_n, $\tilde{r}_n$ and d_n. The main tool is the following lemma. Recall that Ω is called regular if, for each $\zeta \in \Omega$,

$$\lim_{z\to\partial\Omega} g(z, \zeta) = 0.$$

It follows from Harnack's inequality that this then holds uniformly for ζ belonging to any closed subset of Ω.

Lemma 1. *Suppose Ω is regular. Then given $\epsilon > 0$ we can find an open set Ω_1, satisfying $K \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$ and whose boundary is a finite union of rectifiable Jordan curves, such that for each sufficiently large n there is a rational function R, of degree n and having all poles outside Ω, for which*

$$|R(z)| \geqslant e^{-\epsilon n}, \qquad z \notin \Omega_1$$

$$|R(z)| \leqslant e^{-n/C(K,\Omega)+\epsilon}, \qquad z \in K.$$

Proof. Let K_1 be a closed subset of Ω whose interior contains K and such that

$$C(K_1, \Omega) \leqslant C(K, \Omega) + \epsilon. \tag{5}$$

That such a K_1 exists follows easily from (c). Denote by μ_1 the equilibrium measure for K_1 as in (a), so that

$$\mu_1(K_1) = 1, \qquad \int g(z, \zeta)\, d\mu_1(\zeta) \leqslant 1/C(K_1, \Omega)$$

and, by (b),

$$\int g(z, \zeta)\, d\mu_1(\zeta) = 1/C(K_1, \Omega), \qquad z \in K. \tag{6}$$

By the regularity of Ω, we can find Ω_1 satisfying $K_1 \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$, and whose boundary is a finite union of rectifiable Jordan curves, such that

$$g(z, \zeta) \leqslant \epsilon, \qquad \zeta \in \partial K_1, \qquad z \in \partial \Omega_1. \tag{7}$$

We know from (b) that μ_1 is supported on ∂K_1. Moreover, it is the weak limit of discrete measures. Therefore, we can find a finite set of points $\zeta_i \in \partial K_1$, and constants α_i satisfying

$$\alpha_i \geqslant 0, \qquad \sum \alpha_i = 1, \tag{8}$$

such that

$$\left| \int g(z, \zeta)\, d\mu_1(\zeta) - \sum \alpha_i g(z, \zeta_i) \right| \leqslant \epsilon, \qquad z \in K. \tag{9}$$

If h_ζ denotes harmonic measure on $\partial \Omega$ then $g(z, \zeta)$ has the representation

$$g(z, \zeta) = \log \frac{1}{|z - \zeta|} - \int \log \frac{1}{|z - \xi|}\, dh_\zeta(\xi) + g(\infty, \zeta).$$

(This particular form of the representation assumes $\infty \in \Omega$, which can always be achieved by a linear fractional transformation.) If $\{\Xi_j\}$ is a finite partition of $\partial \Omega$ which is sufficiently fine, and $\xi_j \in \Xi_j$, then we shall have

$$\left| g(z, \zeta) - \left\{ \log \frac{1}{|z - \zeta|} - \sum_j h_\zeta(\Xi_j) \log \frac{1}{|z - \xi_j|} + g(\infty, \zeta) \right\} \right| < \epsilon,$$
$$\zeta \in \partial K_1, \qquad z \in K \cup \partial \Omega_1. \tag{10}$$

Define

$$\beta_j = \sum_i \alpha_i h_{\zeta_i}(\Xi_j).$$

Then

$$\sum_j \beta_j = \sum_i \alpha_i \sum_j h_{\zeta_i}(\Xi_j) = \sum_i \alpha_i = 1 \tag{11}$$

since for each ζ, $\sum_j h_\zeta(\Xi_j) = h_\zeta(\partial \Omega) = 1$.

Let us now combine (10) with (7), (9), (6), and (5). We obtain

$$\sum \alpha_i \log \frac{1}{|z-\zeta_i|} - \sum \beta_j \log \frac{1}{|z-\xi_j|} + \sum \alpha_i g(\infty, \zeta_i)$$

$$\begin{cases} \leqslant 2\epsilon, & z \in \partial\Omega_1 \\ \geqslant \{C(K, \Omega) + \epsilon\}^{-1} - 2\epsilon, & z \in K. \end{cases}$$

It is a simple exercise, left to the reader, to show that for each integer n we can find numbers a_i, b_j satisfying

$$|a_i - \alpha_i| \leqslant n^{-1}, \qquad |b_j - \beta_j| \leqslant n^{-1}$$

$$na_i \text{ and } nb_j \text{ are all integers}$$

$$\sum a_i = 1, \qquad \sum b_j = 1.$$

(The last uses (8) and (11).) Then if n is sufficiently large, the rational function

$$R(z) = \prod_i (z-\zeta_i)^{n\alpha_i} e^{-n\alpha_i g(\infty,\zeta_i)} \prod_j (z-\xi_j)^{-n\beta_j},$$

which has degree n and poles at the $\xi_j \in \partial\Omega$ will satisfy

$$|R(z)| \geqslant e^{-3\epsilon n}, \qquad z \in \partial\Omega_1$$

$$|R(z)| \leqslant e^{3\epsilon n}\, e^{-n/C(K,\Omega)+\epsilon}, \qquad z \in K.$$

This is almost the statement of the lemma (with a different ϵ, a matter of no importance). What we have yet to do is extend the lower bound on $|R|$ from $\partial\Omega_1$ to the complement of Ω_1. But this is automatic, since all the zeros of R lie in $K_1 \subset \Omega_1$.

THEOREM 2. *We have*

$$\limsup_{n\to\infty} r_n^{1/n} \leqslant e^{-1/C(K,\Omega)}, \qquad \limsup_{n\to\infty} d_n^{1/n} \leqslant e^{-1/C(K,\Omega)}.$$

If Ω is regular,

$$\limsup_{n\to\infty} \tilde{r}_n^{1/n} \leqslant e^{-1/C(\Omega,K)}. \tag{12}$$

Proof. Assume first that Ω is regular and let R be as in the lemma. Take any $f \in A$ and consider

$$F(z) = \frac{R(z)}{2\pi i} \int_{\partial\Omega_1} \frac{f(\zeta)}{R(\zeta)(\zeta - z)}\, d\zeta, \qquad z \in \Omega_1,$$

where $\partial\Omega_1$ is given the usual orientation so that the Cauchy integral formula holds. The estimates on R give immediately

$$|F(z)| \leqslant ce^{\epsilon n}e^{-n/C(K,\Omega)+\epsilon}, \qquad z \in K, \tag{13}$$

where c is a constant depending on the geometry of K and $\partial\Omega_1$.

But $F(z)$ is just $R(z)$ times the sum of the residues at the poles of the integrand. There is a pole at $\zeta = z$ and poles at the ζ_i, the zeros of R inside Ω_1. Consequently, if the multiplicity of ζ_i is m_i then there are constant $c_{i,k}$ $(1 \leqslant k \leqslant m_i)$ such that

$$F(z) = f(z) - R(z) \sum_{i,k} c_{i,k}(z - \zeta_i)^{-k}.$$

The last function on the right is rational and has poles among those of R. It is therefore at most of degree n. In view of (13) we have established the last statement of the theorem (and also the first for regular Ω, since $r_n \leqslant \tilde{r}_n$). Moreover, there are $\sum m_i \leqslant n$ functions

$$R(z)(z - \zeta_i)^{-k}$$

whose linear combinations are the approximants of f. Thus we have also established the second statement of the theorem for regular Ω.

Now assume Ω is arbitrary. We can find an increasing family $\{\Omega_n\}$ of open sets, each regular, whose union is Ω. Since both r_n and d_n are not decreased if one replaces Ω by a smaller open set, an application of fact (d) yields the first two statements of the theorem.

4. Limiting Behavior of s_n

THEOREM 3. (4) *holds.*

Proof. In view of (3) we need only establish the lower bound

$$\liminf_{n\to\infty} s_n^{1/n} \geqslant e^{1/C(K_2, K_1')}.$$

Let Ω be a regular open set contained in K_1' and satisfying

$$C(K_2, \Omega) \leqslant C(K_2, K_1') + \epsilon$$

(such an Ω exists by (d)) and apply Lemma 1 with K_2 as K. Then $|R| \geqslant e^{-\epsilon n}$ outside Ω_1 and so in K_1, and

$$|R| \leqslant e^{-n/C(K_2, K_1')+2\epsilon}$$

in K_2. The result follows.

5. Lower Bound for d_n

Let μ be the equilibrium measure on K satisfying

$$\mu(K) = 1, \qquad u_\mu(z) = \int g(z, \zeta)\, d\mu(\zeta) \leqslant 1/C(K, \Omega).$$

Let Ω_1 be any open set satisfying

$$K \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$$

and whose boundary Γ is a finite union of rectifiable Jordan curves, oriented in the usual way for the Cauchy integral formula to hold in $\bar{\Omega}_1$.

Lemma 4. *If $\tilde{u}_\mu$ is the (multiple valued) harmonic conjugate of u_μ, then*

$$\mathop{\Delta}_{z \in \Gamma} \tilde{u}_\mu(z) = -2\pi.$$

Proof. Take any $\zeta \in K$ and let D be a disc containing ζ and contained in Ω_1. Since $g(z, \zeta)$ is harmonic for $z \in \Omega_1 - D$,

$$\mathop{\Delta}_{\Gamma} \tilde{g}(z, \zeta) = \mathop{\Delta}_{\partial D} \tilde{g}(z, \zeta).$$

Since $g(z, \zeta) + \log | z - \zeta |$ is harmonic in D

$$\mathop{\Delta}_{\partial D} \tilde{g}(z, \zeta) = - \mathop{\Delta}_{\partial D} \arg(z - \zeta) = -2\pi.$$

Hence

$$\mathop{\Delta}_{z \in \Gamma} \tilde{u}_\mu(z) = \int \mathop{\Delta}_{z \in \Gamma} \tilde{g}(z, \zeta)\, d\mu(\zeta) = -2\pi.$$

Remark. $C(K, \Omega)\, u_\mu(z)$ is equal, in Ω–K, to the harmonic measure h of K with respect to Ω–K. Therefore the lemma gives

$$\mathop{\Delta}_{z \in \Gamma} \tilde{h}(z) = -C(K, \Omega).$$

But since $1 - h$ is the harmonic measure of Ω' with respect to Ω–$K = K'$–Ω', and since $\bar{\Omega}_1'$ serves for Ω', K' as Ω_1 does for K, Ω, we deduce

$$\mathop{\Delta}_{z \in \Gamma} (1 - \tilde{h}(z)) = -C(\Omega', K'),$$

where here Γ is described in the opposite direction. Thus

$$C(K, \Omega) = C(\Omega', K'),$$

the relation asserted in section 2.

LEMMA 5. *Suppose Ω is a connected open set with finitely many boundary components none of which is a point, or is a finite disjoint union of such sets. Let $\gamma_1, \ldots, \gamma_n$ be a homology basis for Ω. Then there is a constant $M > 0$ such that for any real numbers $\alpha_1, \ldots, \alpha_n$ there is a harmonic function h defined in Ω and satisfying*

$$0 \leqslant h(z) \leqslant M,$$

$$\underset{z \in \gamma_i}{\Delta} \tilde{h}(z) \equiv \alpha_i \pmod{2\pi}.$$

Proof. We may assume first that Ω is connected, and second, since the problem is conformally invariant, that Ω is the region interior to a Jordan curve C_0 and exterior to Jordan curves $C_1, \ldots, C_n$. If the result holds for one homology basis it holds for all, so we may assume further that each γ_i is homologous to C_i. Choose ζ_i interior to C_i and set

$$m_i = \operatorname{dist}(\zeta_i, C_i).$$

For any α_i, determine β_i by

$$0 \leqslant \beta_i < 1, \qquad 2\pi\beta_i \equiv \alpha_i \pmod{2\pi},$$

and set

$$h(z) = \sum \beta_i \{\log | z - \zeta_i | - \log m_i\}.$$

Then h satisfies the conditions of the lemma and has an upper bound M independent of the α_i.

The lower bound on d_n is obtained by constructing many functions in A which are not too much larger on Ω than on K. We start with

$$e^{-n\{u_\mu(z) + i\tilde{u}_\mu(z)\}}$$

which is

$$e^{n/C(K,\Omega)}$$

times as large on $\partial\Omega$ as on K. It is, however, not in A for two reasons: it is not single-valued, and it is not analytic on K. The first difficulty is easily dealt with by Lemma 5. We assume Ω is of the form described in the statement of the lemma. Then we can find a multiple-valued function

$$\psi(z) = e^{-(h(z) + i\tilde{h}(z))}$$

such that

$$e^{-M} \leqslant | \psi(z)| \leqslant 1$$

and such that

$$\varphi(z) = \psi(z)\, e^{-n\{u_\mu(z)+i\tilde{u}_\mu(z)\}}$$

is single-valued in $\Omega - K$.

The modification of this function to one analytic also on K, without changing its crucial characteristic property of being not much larger on Ω than on K, is achieved by solving a certain Riemann boundary value problem. The open set Ω_1 is as defined at the beginning of this section. We take Ω_2 to be the complement of $\bar{\Omega}_1$, and as before Γ is the common boundary of Ω_1 and Ω_2.

LEMMA 6. *Suppose we can find functions f_i $(i = 1, 2)$, analytic in Ω_i and continuous on $\bar{\Omega}_i$, whose boundary values satisfy*

$$f_1(z) = \varphi(z) f_2(z), \qquad z \in \Gamma. \tag{14}$$

Then the function

$$f(z) = \begin{cases} f_1(z), & z \in \Omega_1 \\ \varphi(z) f_2(z), & z \in \bar{\Omega}_2 \cap \Omega \end{cases}$$

is analytic in Ω and satisfies

$$\sup_{z \in \Omega} |f(z)| \leqslant e^M e^{n/C(\Omega,K)} \max_{z \in K} |f(z)|.$$

Proof. The analyticity of f follows immediately from the analyticity of its constituent parts and its continuity on Γ. We have

$$\sup_{z \in \Omega} |f(z)| = \limsup_{z \to \partial\Omega} |f(z)|,$$

and since Ω_2 is a neighborhood of $\partial\Omega$ this is at most

$$\limsup_{z \to \partial\Omega} |f_2(z)|.$$

(Note that $|\varphi| \leqslant 1$.) Next consider the function

$$g(z) = \begin{cases} f_1(z)/\varphi(z), & z \in \Omega_1 - K, \\ f_2(z), & z \in \bar{\Omega}_2, \end{cases}$$

which is analytic in K'. At every regular point of ∂K the function φ has lim sup at most

$$e^M\, e^{n/C(K,\Omega)}.$$

Thus $|g(z)|$ is bounded in K' and at all regular points of ∂K has lim sup at most

$$e^M e^{n/C(K,\Omega)} \max_{z\in K} |f_1(z)|.$$

Consequently, by the maximum principle for subharmonic functions (see [7, Theorems III. 33 and III. 28]),

$$\sup_{z\notin K} |g(z)| \leqslant e^M e^{n/C(K,\Omega)} \max_{z\in K} |f_1(z)|.$$

In particular,

$$\limsup_{z\to\partial\Omega} |f_2(z)| \leqslant e^M e^{n/C(K,\Omega)} \max_{z\in K} |f(z)|,$$

and this gives the desired result.

We now give a sharp lower bound for d_n in certain cases.

THEOREM 7. *Suppose Ω is a connected open set with finitely many boundary components, or is a finite disjoint union of such sets. Then for some constant $a > 0$ we have*

$$d_n \geqslant a\, e^{-n/C(K,\Omega)}.$$

Proof. Since adding an isolated boundary point to Ω changes neither d_n (by the theorem on removable singularities) nor $C(K, \Omega)$, we may assume Ω satisfies the hypothesis of Lemma 5.

It is well known that the dimension of the (linear) space E of solutions of (14) is

$$\begin{aligned} N &= \frac{1}{2\pi} \mathop{\Delta}_{z\in\Gamma} \arg \varphi(z) \\ &= \frac{1}{2\pi} \mathop{\Delta}_{z\in\Gamma} \arg \psi(z) - \frac{n}{2\pi} \mathop{\Delta}_{z\in\Gamma} \tilde{u}_\mu(z). \end{aligned} \tag{15}$$

See, for example, [3, section 16. 2], where (14) is shown for connected Ω_1; the general case is an easy consequence.

Since ψ is bounded and bounded away from zero, uniformly in n,

$$\mathop{\Delta}_{z\in\Gamma} \arg \psi(z)$$

is bounded. Therefore an application of Lemma 4 gives

$$N = n + O(1). \tag{16}$$

Now let E_{N-1} be any linear space of continuous functions on K. Then since E has dimension N, we can find an $f \in E$ satisfying

$$\max_{z \in K} |f(z)| \leqslant 1, \qquad \operatorname{dist}(f, E_{N-1}) = 1.$$

(See [6, p. 137].) If we apply Lemma 6 we find that the function

$$e^{-M} e^{-n/C(K,\Omega)} f(z)$$

belongs to A and has distance to E_{N-1} at least

$$e^{-M}\, e^{-n/C(K,\Omega)}.$$

Therefore,

$$d_{N-1} \geqslant e^{-M}\, e^{-n/C(K,\Omega)},$$

and in view of (16) this establishes the theorem.

Remark. It is clear that the theorem holds for any open set for which the analogue of Lemma 4 is valid. It is a fact that for any open set we can find a harmonic function with conjugate having given periods, and so certainly periods in a given congruence class, but when bounds are needed it is another matter. There are many infinitely connected domains for which the required functions can be found, but it is not true for all domains. For example, one can show that it fails if Ω is the complement of a Cantor set.

We now state a weaker inequality in a more general situation.

THEOREM 8. *Suppose the complement of Ω has countably many (connected) components. Then*

$$\liminf_{n \to \infty} d_n^{1/n} \geqslant e^{-1/C(K,\Omega)}.$$

Proof. Suppose the components of Ω are the (open) sets $\Omega_1, \Omega_2, \ldots$, necessarily countable in number. Finitely many of the Ω_i, say $\Omega_1, \ldots, \Omega_n$, cover K. Clearly d_n for K and Ω is the same as d_n for K and $\Omega_1 \cup \cdots \cup \Omega_n$, and

$$C(K, \Omega) = C(K, \Omega_1 \cup \cdots \cup \Omega_n).$$

Moreover, any component of $(\Omega_1 \cup \cdots \cup \Omega_n)'$ contains either a component of Ω' or one of the sets $\Omega_{n+1}, \Omega_{n+2}, \ldots$ It follows that $(\Omega_1 \cup \cdots \cup \Omega_n)'$ can have only countably many components. Thus we may assume to begin with that $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$.

Denote the components of Ω' by $C_1, C_2, \ldots$, and set

$$\Omega^{(m)} = (C_1 \cup \cdots \cup C_m)'.$$

Since

$$\Omega^{(m)} = \Omega_1 \cup \cdots \cup \Omega_n \cup C_{m+1} \cup C_{m+2} \cup \cdots$$

and each C_j meets some $\bar{\Omega}_i$ $(1 \leqslant i \leqslant n)$, each component of $\Omega^{(m)}$ contains some Ω_i. There are therefore only finitely many (at most n) such components. Thus both $\Omega^{(m)}$ and its complement have finitely many components, so $\Omega^{(m)}$ satisfies the conditions of Theorem 7.

Now as $m \to \infty$ the $\Omega^{(m)}$ decrease and have intersection Ω. Since d_n increases with decreasing Ω, the statement of the theorem follows from fact (d) and Theorem 7.

6. Lower Bound for r_n

The idea is really not much different from [5, Section 6], where it was assumed that K consisted of Jordan curves. It is a matter essentially of applying Rouché's theorem to prove that if a certain approximation is very close on K then the approximating rational function has a large number of zeros and is therefore of large degree. However, in the more general case K may have no interior and we first have to deduce approximations near K from those on K, and this is slightly complicated technically.

We shall denote by $G(z, \zeta)$ Green's function for K', the complement of K. For any positive Borel measure ν on K' we write

$$v_\nu(z) = \int G(z, \zeta)\, d\nu(\zeta).$$

Lemma 9. *Suppose K has finitely many components none of which is a single point, let U be an open set containing K, and let $\epsilon > 0$. Then there is a $\delta > 0$ with the following property: for any ν satisfying $\nu(K') = 1$ there is an open set V satisfying*

$$K \subset V \subset \bar{V} \subset U,$$

$$\operatorname{dist}(K, \partial V) \geqslant \delta,$$

$$\sup_{z \in \partial V} v_\nu(z) \leqslant \epsilon.$$

Proof. We may clearly suppose K is itself connected, and then, since the situation is conformally invariant, we may even assume K is the closed unit disc. Green's function for K' is

$$G(z, \zeta) = \log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right|. \tag{17}$$

We suppose that U contains the disc

$$|z| \leqslant r \qquad (r > 1).$$

Define

$$v_n(z) = \int \max\{G(z, \zeta), n\}\, d\nu(\zeta),$$

so that v_n is continuous and

$$v_n(z) \nearrow v(z).$$

Let

$$S_n = \{z : v_n(z) \geqslant \epsilon\} \cap \{z : 1 < |z| \leqslant r\}.$$

We concentrate first on the nontrivial case when S_n is nonempty. Observe that since v_n is continuous S_n is relatively closed in K'.

Let $S_n{}^*$ denote the set of radii ρ for which the circle $|z| = \rho$ intersects S_n, and let δ_n be the smallest number such that

$$[1 + \delta_n, r] \subset S_n{}^*.$$

We shall find a positive lower bound on δ_n, independent of n and of ν.

Let μ be the equilibrium measure for the Green capacity $C(S_n, K')$, so that μ is supported on S_n,

$$\mu(S_n) = 1, \qquad \int G(z, \zeta)\, d\mu(z) \leqslant 1/C(S_n, K').$$

Then we have

$$\begin{aligned}\epsilon &\leqslant \int v_n(z)\, d\mu(z) \leqslant \int d\mu(z) \int G(z, \zeta)\, d\nu(\zeta)\\ &= \int d\nu(\zeta) \int G(z, \zeta)\, d\mu(z) \leqslant 1/C(S_n, K').\end{aligned}$$

Therefore,

$$C(S_n, K') \leqslant 1/\epsilon. \tag{18}$$

Let

$$A_k = S_n \cap \{z : 1 + e^{-2k} \leqslant |z| \leqslant 1 + e^{-2k+1}\} \qquad k = 1, 2, \ldots$$

The full interval

$$[1 + e^{-2k}, 1 + e^{-2k+1}]$$

belongs to A_k^* (this has the relation to A_k that S_n^* does to S_n) if

$$k_0 \leqslant k \leqslant k_1,$$

where

$$\begin{aligned} k_0 &= \tfrac{1}{2}(1 - \log r), \\ k_1 &= \tfrac{1}{2} \log \delta_n^{-1}. \end{aligned} \tag{19}$$

k_0 is fixed and we want an upper bound on k_1.

Now the ordinary logarithmic capacity of A_k is at least $\frac{1}{4}$ of the linear measure of A_k^* [7, Corollary 6, p. 85], and this is at least $a_1 e^{-2k}$ where a_1 is a positive constant. Hence we can find a measure μ_k on A_k satisfying

$$\mu_k(A_k) = 1, \qquad \int \log \frac{1}{|z - \zeta|} \, d\mu_k(\zeta) \leqslant 2k + \log a_1^{-1}.$$

It follows [see (17)] that

$$\iint G(z, \zeta) \, d\mu_k(\zeta) \, d\mu_k(z) \leqslant 2k + a_2 \tag{20}$$

for another constant a_2.

If $z \in A_k$, $\zeta \in A_{k+j}$ $(j > 0)$, then

$$|z| \geqslant 1 + e^{-2k}, \qquad |\zeta| \leqslant 1 + e^{-2(k+j)+1},$$

$$|\zeta - \bar{\zeta}^{-1}| = |\zeta| - |\zeta|^{-1} \leqslant 2e^{-2(k+j)+1}.$$

Since

$$\frac{\bar{\zeta}z - 1}{z - \zeta} = \bar{\zeta}\left[1 + \frac{\zeta - \bar{\zeta}^{-1}}{z - \zeta}\right]$$

we have

$$\begin{aligned} G(z, \zeta) &\leqslant e^{-2(k+j)+1} + \frac{2e^{-2(k+j)+1}}{e^{-2k} - e^{-2(k+j)+1}} \\ &\leqslant e^{-2j} + \frac{2e^{-2j+1}}{1 - e^{-2j+1}}. \end{aligned}$$

Therefore

$$G(z, \zeta) \leqslant a_3 \, e^{-2j}, \qquad z \in A_k, \qquad \zeta \in A_{k+j}. \tag{21}$$

Let μ be the measure

$$\mu = \sum_{k=k_0}^{k_1} k^{-1}\mu_k$$

on S_n. Then

$$\mu(S_n) \geqslant \log k_1 - a_4$$

and

$$\iint G(z, \zeta)\, d\mu(\zeta)\, d\mu(z) = \sum_{k=k_0}^{k_1} k^{-2} \iint G(z, \zeta)\, d\mu_k(\zeta)\, d\mu_k(z)$$

$$+ 2 \sum_{k=k_0}^{k_1} k^{-1} \sum_{j=1}^{k_1-k} \iint G(z, \zeta)\, d\mu_{k+j}(\zeta)\, d\mu_k(\zeta).$$

By (20) and (21) this is at most

$$a_5 \log k_1 + a_6 .$$

But since $C(S_n, K')$ is equal to the least upper bound of the ratios

$$\mu(S_n)^2 \Big/ \iint G(z, \zeta)\, d\mu(\zeta)\, d\mu(z)$$

taken over all μ supported on S_n, we deduce

$$C(S_n, K) \geqslant a_7 \log k_1 - a_8 .$$

Combining this with (18) and (19) gives

$$\log \log \delta_n^{-1} \leqslant a_9 \epsilon^{-1}.$$

It follows from this bound, and the way δ_n was defined, that for some constant $\delta > 0$ depending only on ϵ there is for each n a ρ_n in

$$1 + \delta \leqslant \rho_n \leqslant r$$

such that

$$v_n(z) < \epsilon \qquad \text{on} \quad |z| = \rho_n .$$

This was derived under the assumption that S_n was nonempty. But if S_n is empty we simply take $\rho_n = r$. An application of Fatou's lemma shows that

$$v(z) \leqslant \epsilon \qquad \text{on} \quad |z| = \rho$$

for any limit point ρ of the sequence $\{\rho_n\}$. Thus we may take V to be any one of the discs $|z| < \rho$ and the lemma is established.

THEOREM 10. *Suppose K and the complement of Ω have countably many connected components. Then*

$$\liminf_{n\to\infty} r_n^{1/n} \geqslant e^{-1/C(K,\Omega)}.$$

Proof. We consider only the case where K and Ω' have finitely many components, none a single point. The general result is deduced as in the proof of Theorem 8.

The harmonic measure of $\partial\Omega$ in Ω–K is at most ϵ in some neighborhood U of K. This implies, by the maximum principle, that if $\sigma(z)$ is subharmonic in Ω–K and satisfies

$$\sigma(z) \leqslant 0$$

there, then

$$\sigma(z) \leqslant (1-\epsilon) \limsup_{\zeta\to\partial K} \sigma(\zeta), \qquad z \in U - K. \tag{22}$$

Having found U, we let δ be as in Lemma 9, and set

$$W = \{z \in U : \operatorname{dist}(z, K) \geqslant \delta\}.$$

What follows next is very similar to the construction of Lemma 1. Let u_μ once again denote the equilibrium potential for K, so that

$$\mu(K) = 1, \qquad u_\mu(z) = \int g(z, \zeta)\, d\mu(\zeta) \leqslant 1/C(K, \Omega).$$

Since $g(z, \zeta)$ is continuous for $z \in W$, $\zeta \in K$ we can find points $\zeta_i \in K$ and numbers α_i satisfying

$$\alpha_i > 0, \qquad \Sigma\, \alpha_i = 1$$

such that

$$\sum \alpha_i g(z, \zeta_i) < 1/C(K, \Omega) + \epsilon, \qquad z \in W.$$

Next, given $n > 0$ we can find numbers a_i satisfying

$$|a_i - \alpha_i| \leqslant n^{-1}, \qquad \sum a_i = 1$$

such that each na_i is an integer. Then for large enough n we shall have

$$\sum a_i g(z, \zeta_i) < 1/C(K, \Omega) + 2\epsilon, \qquad z \in W.$$

If we write

$$h(z) = \sum a_i g(z, \zeta_i)$$

then it follows from Lemma 5 that we can find

$$\psi(z) = e^{-\{h(z)+i\tilde{h}(z)\}}$$

satisfying

$$e^{-M} \leqslant |\psi(z)| \leqslant 1$$

such that

$$f(z) = \psi(z)\, e^{-n\{u_\mu(z)+i\tilde{u}_\mu(z)\}}$$

is single-valued in Ω, and so belongs to A. We shall find a lower bound on $r_{n-1}(f)$

Let R be a rational function of degree $\leqslant n-1$ such that

$$\max_{z\in K} |f(z) - R(z)| = r_{n-1}(f).$$

Denote by ξ_j the poles of R (all lying outside K) and by v the potential

$$v(z) = \sum G(z, \xi_j).$$

The corresponding measure ν, giving measure 1 to each $\{\xi_j\}$, satisfies

$$\nu(K') < n.$$

Let us estimate

$$|f(z) - R(z)|\, e^{-v(z)}.$$

We have, first,

$$|f(z)|\, e^{-v(z)} \leqslant 1, \qquad z \in \Omega\text{–}K.$$

Second,

$$|R(z)|\, e^{-v(z)}$$

is subharmonic in K' and so

$$|R(z)|\, e^{-v(z)} \leqslant \max_{z\in K} |R(z)| \leqslant 2, \qquad z \in K',$$

if, as we may certainly assume, $r_{n-1}(f) \leqslant 1$. Hence

$$|f(z) - R(z)|\, e^{-v(z)} \leqslant 3, \qquad z \in \Omega - K.$$

It follows from (22) that consequently

$$|f(z) - R(z)|\, e^{-v(z)} \leqslant 3r_{n-1}(f)^{1-\epsilon}, \qquad z \in U - K.$$

In particular this holds for $z \in W$.

Let us now apply Lemma 9. There is an open set V, containing K and with $\partial V \subset W$, for which

$$\sup_{z \in \partial V} v(z) \leqslant \epsilon n.$$

Then

$$\max_{z \in \partial V} |f(z) - R(z)| \leqslant 3e^{\epsilon n} r_{n-1}(f)^{1-\epsilon}.$$

However, R has at most $n - 1$ zeros in V and f has zeros in V of total multiplicity

$$\sum na_i = n.$$

Therefore by Rouché's theorem the above maximum is at least equal to

$$\min_{z \in \partial V} |f(z)| \geqslant e^{-M} e^{-n/C(K,\Omega)+2\epsilon}.$$

Since $\epsilon > 0$ was arbitrary the theorem is proved.

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